Off-Shell Formulation of Simple Supersymmetric Yang-Mills

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Abstract

An off-shell formulation for 6 and 10 dimensions simple supersymmetric Yang-Mills theories is presented. While the fermionic fields couple to left action of S^3 and S^7 respectively, the auxiliary ones couple to right action (and vice versa). To close the algebra off-shell, left and right actions must commute. For 6 dimensions quaternions work fine. The 10 dimensional case needs special care. Pure spinors and soft Lie algebra (algebra with structure functions instead of structure constants) are essential. Some tools useful for constructing the superspace are also derived. We show how to relate our results to the early works of Evans and Berkovits.

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1 Introduction

Day after day, supersymmetry consolidates its position in theoretical physics. Even if it was introduced more than 25 years ago, there are still problems with the geometric basis of extended ($N > 1$) supersymmetry. The situation of the extended superspace is far less satisfactory than the original N=1 superspace. At the level of the algebra the on-shell formalism closes up to modulo of the classical equations of motion. This fact seems odd at the quantum level since the equations of motion receive loop corrections[1]. The superspace introduces an elegant supermanifold with different enlarged superconnections, where some are truly integrable in the sense of having zero supercurvature. In principle, the extended superspace should be a very powerful tool for quantum calculations.

Before starting, we feel obliged to mention something about the history of the following conjecture: Ring Division Algebras $\mathbb{K} \equiv \{$ real $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$, octonions $\mathbb{O}$ $\}$ are relevant to simple supersymmetric Yang-Mills. The first hint, as mentioned by Schwarz [2] comes from the number of propagating Bose and Fermi degrees of freedom which is one for $d = 3$, two for $d = 4$, four for $d = 6$ and eight for $d = 10$ suggesting a correspondence with real $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Kugo and Townsend [3] investigated in detail the relationship between $\mathbb{K}$ and the irreducible spinorial representation of the Lorentz group in $d = 3, 4, 6, 10$, building upon the following chain of isomorphisms

$$
\begin{aligned}
so\,(2,1) &\iff sl(2,\mathbb{R}) \\
so(3,1) &\iff sl(2,\mathbb{C}) \\
so(5,1) &\iff sl(2,\mathbb{H}).
\end{aligned}
$$

They conjectured that $so(9,1) \iff sl(2,\mathbb{O})$, the correct relation turned out to be

$$
so(9,1) \iff sl(2,\mathbb{O}) \oplus G_2
$$

as has been shown by Chung and Sudbery [4], i.e. the dimension of $Sl\,(2,\mathbb{O})$ is 31. Also in [3], a quaternionic treatment of the $d = 6$ case is presented. Later, Evans made a systematic investigation of the relationship between

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[1]Also, the supersymmetry transformations receive corrections and one should test the closure of the algebra order by order in perturbation theory.

SSYM and ring division algebra in a couple of papers. In the first [5], he simplified the construction of SSYM by proving a very important identity between gamma matrices by using the intrinsic triality of ring division algebra instead of the "tour de force" used originally by Brink, Scherck and Schwarz [1] via Fierz identities generalized to $d > 4$ dimensions. Then, in the second paper [6], Evans made the connection even clearer by showing how the auxiliary fields are really related to ring division algebras. For $d = 3, 4, 6, 10$ we need $k = 0, 1, 3, 7$ auxiliary fields respectively. An alternative approach for the octonionic case was introduced by Berkovits [7] who invented a larger supersymmetric transformation called generalized supersymmetry in [8]. There has also been a twistor attempt by Bengtsson and Cederwall [9]. For more references about the octonionic case and ten dimensional physics one may consult references in [10] and its extension to p-branes by Belecowe and Duff [11]. The early work of Nilsson may be relevant [12][13] too.

As a first step towards an extended superspace, we address the point of the algebraic auxiliary fields for simple N=1 supersymmetric Yang-Mills (SSYM) definable only in $d = 3, 4, 6$ and 10 dimensions [1]. The important point is: *While the physical fields couple to ring division left action the auxiliary ones couple to right action (or vice versa)*. To admit a closed off-shell supersymmetric algebra, left and right action must commute i.e. we should have a parallelizable associative algebra. For d = 6, quaternions work fine but for d = 10, the only associative seven dimensional algebra that is known is the soft seven sphere. We shall show below how this works. In this work, we use the same symbols (left action $\equiv \mathbb{E}_j$, right action $\equiv 1|\mathbb{E}_j$) for either complex, quaternionic or octonionic numbers and each case should be distinguished by the range of the indices j which run from 1 to $(1, 3, 7)$ for complex, quaternions and octonions respectively.

In the second section, we review the relation between hypercomplex structure and Clifford algebra. The auxliary fields problem in 6 dimensions is presented in the third section. While section four is devoted to the ten dimensional case. The last section contains some superspace hints.

2 Hypercomplex Structure and Pure Spinors

Everything starts from Clifford algebra, so let's review quickly the connection between hypercomplex structures and our gamma matrices in $d = 3, 4, 6, 10$.

The solution is encoded completely in our Γ_M. Algebraically, we can construct a Clifford algebra directly from complex, quaternions and octonions over $1, 3, 7$ Euclidean space which can be extended easily to a representation of the minimal irreducible spinorial subspaces in $d = 3, 4, 6$ and 10 Minkowskian space-time. Consider the set of matrices $\{\mathbb{E}_j\}$ for the following three different cases[14][15]:

- the canonical complex structure over $\mathbb{R}^2$ is just 2×2 matrix $\mathbb{E}_1$

$$e_1 \Longleftrightarrow \mathbb{E}_1 = (\delta_{0\mu}\delta_{1\nu} - \delta_{0\nu}\delta_{1\mu}); \quad (\mathbb{E}_1) = -\mathbf{1}_\mu \quad ; \quad \mu = 0, 1 \quad , \quad (1)$$

by $\mathbf{1}_\mu$ we always mean an $(\mu \times \mu)$ unit matrix.

- • the canonical left quaternionic structures over $\mathbb{R}^4$ are

$$(\mathbb{E}_j)_{\mu\nu} = (\delta_{0\mu}\delta_{j\nu} - \delta_{0\nu}\delta_{j\mu} - \epsilon_{j\mu\nu}) \qquad \mu, \nu = 0..3; \; j, k, h = 1..3 \quad , \quad (2)$$

and

$$\mathbb{E}_j\mathbb{E}_k = (-\delta_{jk}\mathbf{1}_\mu + \epsilon_{jkh}\mathbb{E}_h) \qquad , \tag{3}$$

where ϵ_{jkh} is the standard Levi-Civita symbol. Using Rotelli's notation for right action[16], the canonical right quaternionic structures are,

$$(1|\mathbb{E}_j)_{\mu\nu} = (\delta_{0\mu}\delta_{j\nu} - \delta_{0\nu}\delta_{j\mu} + \epsilon_{j\mu\nu}) \quad , \tag{4}$$

and

$$1|\mathbb{E}_j \; 1|\mathbb{E}_k = (-\delta_{jk}\mathbf{1}_\mu - \epsilon_{jkh}1|\mathbb{E}_h) \qquad . \tag{5}$$

Let's put these quaternionic structures into a form that can be recognized by physicsits

$$(\mathbb{E}_j)_{\mu\nu} = -(1|\mathbb{E}_j)_{\mu\nu} = -\epsilon_{j\mu\nu} \quad if \quad \mu, \nu = 1, 2, 3. \quad (\mathbb{E}_j)_{00} = (1|\mathbb{E}_j)_{00} = 0 \quad .$$

$$(\mathbb{E}_j)_{0\nu} = (1|\mathbb{E}_j)_{0\nu} = -\delta_{j\nu}, \qquad (\mathbb{E}_j)_{\mu 0} = (1|\mathbb{E}_j)_{\mu 0} = \delta_{j\mu} \quad , \tag{6}$$

such mathematical quaternionic structures are well known in physics as the 't Hooft eta symbols[17]. We can check that

$$\{\mathbb{E}_i, \mathbb{E}_j\} = \{1|\mathbb{E}_i, 1|\mathbb{E}_i\} = -2\delta_{ij}\mathbf{1}_4 \quad ,$$

$$\begin{aligned} [\mathbb{E}_j, \mathbb{E}_k] &= \epsilon_{jkh}\mathbb{E}_h \qquad , \\ [1|\mathbb{E}_j, 1|\mathbb{E}_k] &= -\epsilon_{jkh}1|\mathbb{E}_h \qquad , \end{aligned} \tag{7}$$

4

and the very important formula

$$[\mathbb{E}_i, 1|\mathbb{E}_j] = 0 \quad , \tag{8}$$

i.e. left and right quaternionic actions commute.

For octonions, the story is quite different, as they are non-associative. But as it is well known, for any Lie algebra the structure constants are proportional to the constant torsion over the group manifold whereas the torsion over the seven sphere S^7 varies from one point to another [18]. The only way to solve these problems is to use the S^7 as an associative soft Lie algebra[2] as had been proposed by Englert, Servin, Troost, Van Proeyen and Spindel [20] which can be derived from octonions (Look to [21] for a full algebraic investigation of the soft seven sphere). For a generic octonionic number,

$$\varphi = \varphi^\mu e_\mu = \varphi_0 e_0 + \varphi_i e_i, \iff \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \begin{matrix} \mu, \nu = 0..7 \quad , \qquad j, k, h = 1..7 \\ \\ \varphi^\mu \in \mathbb{R} \end{matrix} \quad , \tag{9}$$

such that $e_0 = 1$ and the other seven imaginary units satisfy $e_i e_j = -\delta_{ij} + f_{ijk} e_k \iff [e_i, e_j] = 2f_{ijk} e_k$ where f_{ijk} is completely antisymmetric and equals one for any of the following three-cycles (123), (145), (246), (347), (176), (257), (365). To construct the soft seven sphere Lie algebra, we just have to define the direction of action, for left and right action, we have

$$\begin{aligned} \delta_i \varphi &= e_i \varphi \quad , \\ 1|\delta_i \varphi &= \varphi e_i \quad , \end{aligned} \tag{10}$$

then after simple calculations, we find

$$[\delta_j, \delta_k] = 2f_{jkh}\delta_h - 2[\delta_j, 1|\delta_k] \quad ,$$

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[2]Soft Lie algebra is an algebra with structure functions instead of structure constants [19].

$$[1|\delta_j, 1|\delta_k] = -2f_{jkh}1|\delta_h - 2[\delta_j, 1|\delta_k] \quad ,$$
$$\{\delta_j, \delta_k\} = -2\delta_{jk} \quad ,$$
$$\{1|\delta_j, 1|\delta_k\} = -2\delta_{jk} \quad , \tag{11}$$

which are isomorphic to the following set $\{\mathbb{E}_j, 1|\mathbb{E}_j\}$ of 8×8 matrices,

$$\delta_j \iff (\mathbb{E}_j)_{\mu\nu} = \delta_{0\mu}\delta_{j\nu} - \delta_{0\nu}\delta_{j\mu} - f_{j\mu\nu} \quad ,$$
$$1|\delta_j \iff (1|\mathbb{E}_j)_{\mu\nu} = \delta_{0\mu}\delta_{j\nu} - \delta_{0\nu}\delta_{j\mu} + f_{j\mu\nu} \quad , \tag{12}$$

satisfying the algebra [14]

$$[\mathbb{E}_j, \mathbb{E}_k] = 2f_{jkh}\mathbb{E}_h - 2[\mathbb{E}_j, 1|\mathbb{E}_k] \quad ,$$
$$[1|\mathbb{E}_j, 1|\mathbb{E}_k] = -2f_{jkh}1|\mathbb{E}_h - 2[\mathbb{E}_j, 1|\mathbb{E}_k] \quad ,$$
$$\{\mathbb{E}_j, \mathbb{E}_k\} = -2\delta_{jk} \quad ,$$
$$\{1|\mathbb{E}_j, 1|\mathbb{E}_k\} = -2\delta_{jk} \quad , \tag{13}$$

they don't close a Lie algebra but they close a soft Lie algebra defined by

$$[\delta_j, \delta_k]\varphi \equiv 2f_{jkh}^{(+)}(\varphi) \, e_h\varphi \iff [\mathbb{E}_j, \mathbb{E}_k]\varphi = 2f_{jkh}^{(+)}\mathbb{E}_h\varphi \quad ,$$

$$[1|\delta_j, 1|\delta_k]\varphi \equiv 2f_{jkh}^{(-)}(\varphi) \, \varphi e_h \iff [1|\mathbb{E}_j, 1|\mathbb{E}_k] \, \varphi = 2f_{jkh}^{(-)}1|\mathbb{E}_h \, \varphi \quad , \tag{14}$$

where $f_{jkh}^{(\pm)}(\varphi)$ are the left and right parallelizable torsion. One can check that our $\mathbb{E}_i$ defines what Cartan calls pure spinors [22],

$$\varphi^t\mathbb{E}_i\varphi = 0 \tag{15}$$

thus

$$f_{ijk}^{(+)}(\varphi) = \frac{\varphi^t\left(-\mathbb{E}_k\mathbb{E}_i\mathbb{E}_j\right)\varphi}{r^2}. \tag{16}$$

and

$$f_{ijk}^{(-)}(\varphi) = \frac{\varphi^t\left(-1|\mathbb{E}_k \ 1|\mathbb{E}_i \ 1|\mathbb{E}_j\right)\varphi}{r^2}. \tag{17}$$

where

$$\varphi^t\varphi = r^2. \tag{18}$$

There is another interesting and very important property to note

$$\varphi^t\left[\mathbb{E}_i, 1|\mathbb{E}_j\right]\varphi = 0 \tag{19}$$

which may be the generalization of the standard Lie algebra relation, left and right action commute everywhere over the group manifold.

We close the algebra pointwisely using structure functions $f_{ijk}(\varphi)$ instead of structure constants f_{ijk} where φ may be considered as a coordinate system for an internal S^7 manifold not the space-time x and they don't mix

$$\frac{\partial x}{\partial \varphi} = \frac{\partial \varphi}{\partial x} = 0. \tag{20}$$

Apart from the commutation of left and right actions, there are some other useful identities satisfied by our $(\mathbb{E}_j, 1|\mathbb{E}_j)$ quaternionic or octonionic structures, they are

$$
\begin{aligned}
(\mathbb{E}_k)_{\mu\nu}(\mathbb{E}_j)_{\lambda\nu} + (\mathbb{E}_j)_{\mu\nu}(\mathbb{E}_k)_{\lambda\nu} &= 2\delta_{kj}\delta_{\mu\lambda}, \\
(\mathbb{E}_k)_{\mu\nu}(\mathbb{E}_j)_{\mu\lambda} + (\mathbb{E}_j)_{\mu\nu}(\mathbb{E}_k)_{\mu\lambda} &= 2\delta_{kj}\delta_{\nu\lambda}, \\
(\mathbb{E}_k)_{\mu\nu}(\mathbb{E}_k)_{\lambda\zeta} + (\mathbb{E}_k)_{\lambda\nu}(\mathbb{E}_k)_{\mu\zeta} &= 2\delta_{\mu\lambda}\delta_{\nu\zeta},
\end{aligned} \tag{21}
$$

and the same holds equally well for $(1|\mathbb{E}_j)$, as had been noticed by Evans [5], they are direct consequences of the ring division triality.

Now, we have all the needed ingredients to construct our real universal $(\Gamma_M)_{ab}$ matrices with spinorial lower indices a, b of range the double of the μ. For Minkowskian metric of signature $\eta \equiv (-, +, \ldots, +)$, in $d = 3, 4, 6$ and 10, $a, b = 0..2\mu + 1$, for simplicity, we use symmetric Γ_M,

$$
(\Gamma_j)_{ab} = \begin{pmatrix} 0 & \mathbb{E}_j \\ -\mathbb{E}_j & 0 \end{pmatrix}
\qquad\qquad
(1|\Gamma_j)_{ab} = \begin{pmatrix} 0 & 1|\mathbb{E}_\mu \\ -1|\mathbb{E}_j & 0 \end{pmatrix},
$$

$$
(\Gamma_0)_{ab} = \begin{pmatrix} -\mathbf{1}_\mu & 0 \\ 0 & -\mathbf{1}_\mu \end{pmatrix}; \quad (\Gamma_{d-2}) = \begin{pmatrix} 0 & \mathbf{1}_\mu \\ \mathbf{1}_\mu & 0 \end{pmatrix} \quad (\Gamma_{d-1})_{ab} = \begin{pmatrix} \mathbf{1}_\mu & 0 \\ 0 & -\mathbf{1}_\mu \end{pmatrix},
$$

$$\tag{22}$$

The corresponding higher indices $\left(\tilde{\Gamma}\right)^{ab}$'s are

$$
\left(\tilde{\Gamma}_0\right)^{ab} = -\,(\Gamma_0)_{ab} \qquad and \qquad \left(\tilde{\Gamma}\right)^{ab} = (\Gamma)_{ab} \quad . \tag{23}
$$

As a result, we find

$$
\Gamma^M \tilde{\Gamma}^N + \Gamma^N \tilde{\Gamma}^M = 1|\Gamma^M 1|\tilde{\Gamma}^N + 1|\Gamma^N 1|\tilde{\Gamma}^M = 2\eta^{MN}
$$

or in terms of components

$$(\Gamma^M)_{ab}(\tilde{\Gamma}^N)^{bc} + (\Gamma^N)_{ab}(\tilde{\Gamma}^M)^{bc} = (1|\Gamma^M)_{ab}(1|\tilde{\Gamma}^N)^{bc} + (1|\Gamma^N)_{ab}(1|\tilde{\Gamma}^M)^{bc} = 2\eta^{MN}\delta_a^c. \tag{24}$$

Our Γ's satisfy the very important identity [5]

$$\Gamma_{Ma(b}\Gamma^M_{cd)} = 1|\Gamma_{Ma(b}1|\Gamma^M_{cd)} = 0. \tag{25}$$

3   The SSYM's Auxliary Fields

Using Evans ansatz [5], SSYM are composed of: Gauge fields A_M, spinors Ψ^a, $j (= 1..d-3)$ algebraic auxiliary fields K^j. The gauge group indices will be suppressed in the following. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^{MN} + \frac{i}{2}\Psi^t\Gamma^M\nabla_M\Psi + \frac{1}{2}\delta_{ij}K^iK^j, \tag{26}$$

where $\nabla_M \equiv \partial_M + A_M$; $F_{MN} \equiv [\nabla_M, \nabla_N]$ and the Γ are given in (22). The Lagrangian is invariant up to a total derivative iff (25) holds. Our supersymmetry transformations are[3]

$$\begin{aligned}
\delta_\eta A_M &= i\eta\Gamma_M\Psi, \\
\delta_\eta \Psi^\alpha &= \frac{1}{2}F_{MN}(\Gamma_{MN}\,\eta)^\alpha + K^j(\Lambda_j)^\alpha_\beta\,\eta^\beta, \\
\delta_\eta K_j &= i\left(\Gamma^M\nabla_M\Psi\right)_\alpha (\Lambda_j)^\alpha_\beta\,\eta^\beta,
\end{aligned} \tag{27}$$

where Λ_P are some real matrices and Lorentz transformations are generated by $\Gamma_{MN} \equiv \tilde{\Gamma}_{[M}\Gamma_{N]}$. Imposing the closure of the supersymmetry infinitesimal transformations

$$[\delta_\epsilon, \delta_\eta] = 2i\,{}^t\epsilon\Gamma^M\eta\partial_M\ . \tag{28}$$

The closure on A_M yields

$$\Gamma_M\Lambda_j + (\Lambda_j)^t\Gamma_M = 0. \tag{29}$$

In addition to this condition the closure on K^j also requires

$$\Lambda_j\Lambda_h + \Lambda_h\Lambda_j = -2\delta_{jh}. \tag{30}$$

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[3]Contrary to [6], we set $\Lambda_j = \tilde{\Lambda}^j$ from the strart.

8

While closure on the fermionic field Ψ^α holds iff

$$\left(\Gamma^M\right)_{\alpha\beta}\left(\tilde{\Gamma}_M\right)^{\gamma\delta} = 2\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} + 2\left(\Lambda_j\right)^\gamma_{(\alpha}\left(\Lambda_j\right)^\delta_{\beta)}.$$

Now, we continue in a different way to Evans. To construct Λ_j, we first impose the additional condition

$$\left(\Lambda\right)^t = -\left(\Lambda\right), \tag{31}$$

we notice from (30) that the Λ_j form a real Clifford algebra, and from (29)

$$\Gamma_M\Lambda_j - \Lambda_j\Gamma_M = 0. \tag{32}$$

that they commute with our space-time Γ_M Clifford algebra. The solution of the auxiliary field problem for $d = 3, 4, 6$ dimensions, using (22) is then simply

$$\Lambda_j = \begin{pmatrix} 1|\mathbb{E}_j & 0 \\ 0 & 1|\mathbb{E}_j \end{pmatrix}, \tag{33}$$

because

$$\{1|\mathbb{E}_j, 1|\mathbb{E}_h\} = -2\delta_{jh}, \tag{34}$$

and

$$[\mathbb{E}_j, 1|\mathbb{E}_h] = 0. \tag{35}$$

Of course this solution is not unique. For example, if someone had started with $1|\Gamma_M$, he would have found $\Lambda_j = \begin{pmatrix} \mathbb{E}_j & 0 \\ 0 & \mathbb{E}_j \end{pmatrix}$.

Now, we can relax the conditions (22) and (31). In general, one replaces left/right action used for the gamma matrices by right/left action for the Λ_j e.g.

$$(\Gamma_j)_{ab} = \begin{pmatrix} 0 & \mathbb{E}_j|\mathbb{E}_{j+1} \\ -\mathbb{E}_j|\mathbb{E}_{j+1} & 0 \end{pmatrix} \rightarrow (\Lambda_j)_{ab} = \begin{pmatrix} \mathbb{E}_{j+1}|\mathbb{E}_j & 0 \\ 0 & \mathbb{E}_{j+1}|\mathbb{E}_j \end{pmatrix} \tag{36}$$

One writes any Γ and expand it in terms left/right action $(\mathbb{E}_i, 1|\mathbb{E}_j, \mathbb{E}_m|\mathbb{E}_n)$ then the Λ will be given in terms of suitable $(1|\mathbb{E}_{i,}\mathbb{E}_j, \mathbb{E}_n|\mathbb{E}_m)$ taking into account that daigonal elements should be replaced by non-diagonal one and interchanging left/right actions simultaneously.

4   The Ten Dimensions Case

For $d = 10$, working with octonions the situation is different. We know that octonionic left and right action commutes only when applied to φ,

$$\varphi^t \left[\mathbb{E}_j, 1|\mathbb{E}_h\right] \varphi = 0, \tag{37}$$

and φ is just an 8 dimensional column matrix. Up to now, we have not restricted φ by any other conditions. With two different φ, $\left(\varphi^{(1)}, \varphi^{(2)}\right)$, we impose now the conditions that $\varphi^{(i)}$ be fermionic fields. We express our 16 dimensional Grassmanian variables ϵ, η of eqn.(28) in terms of φ,

$$\epsilon = \eta^t$$
$$\Downarrow$$
$$\epsilon = \left(\begin{array}{cc} \varphi^{(1)} & \varphi^{(2)} \end{array}\right); \qquad \eta = \left(\begin{array}{c} \varphi^{(1)} \\ \varphi^{(2)} \end{array}\right) \tag{38}$$

We now rederive (28) for the octonions. The closure conditions of our algebra, *without omitting the Grassmanian variables* are

$$
\begin{aligned}
\eta^t \left(\Gamma_M \Lambda_j - \Lambda_j \Gamma_M\right) \eta &= 0, \\
\eta^t \left(\Lambda_j \Lambda_h + \Lambda_h \Lambda_j\right) \eta &= \eta^t \left(-2\delta_{jh}\right) \eta, \\
\eta^t \left(\left(\Gamma^M\right)_{\alpha\beta} \left(\tilde{\Gamma}_M\right)^{\gamma\delta}\right) \eta &= \eta^t \left(2\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} + 2 \left(\Lambda_j\right)^\gamma_{(\alpha} \left(\Lambda_j\right)^\delta_{\beta)}\right) \eta, \tag{39}
\end{aligned}
$$

which are satisfied for the octonionic representation

$$\left(\Gamma_j\right)_{ab} = \left(\begin{array}{cc} 0 & \mathbb{E}_j \\ -\mathbb{E}_j & 0 \end{array}\right), \qquad \Lambda_j = \left(\begin{array}{cc} 1|\mathbb{E}_j & 0 \\ 0 & 1|\mathbb{E}_j \end{array}\right). \tag{40}$$

By interchanging left/right action, we have different solutions as in the quaternionic case. In summary, while the fermionic fields couple to left/right action through the gamma matrices, the auxiliary fields couple to right/left action through the Λ. For the octonionic case *the presence of the Grassmanian variables is essential.* Contrary to the standard supersymmetry transformation, our Grassman variables are the same ($\epsilon = \eta^t$), which is identical to the result obtained by Berkovits in [7]. According to Evans [8], the attractive feature of this scheme is that the Lagrangian (26) and the transformation

(27) are manifestly invariant under the generalized Lorentz group $SO(1,9)$. In our formulation, we can show some additional characteristic. In some cases, the (38) condition may be relaxed, for equal j or h (no summation)

$$\left. \begin{array}{c} \varphi^t \, \mathbb{E}_j \; [\mathbb{E}_j, 1|\mathbb{E}_h] \, \varphi \\ \varphi^t \; 1|\mathbb{E}_i \; [\mathbb{E}_j, 1|\mathbb{E}_h] \, \varphi \\ \varphi^t \; E_h \; [\mathbb{E}_j, 1|\mathbb{E}_h] \, \varphi \\ \varphi^t \; 1|E_h \; [\mathbb{E}_j, 1|\mathbb{E}_h] \, \varphi \end{array} \right\} = 0. \tag{41}$$

i.e. relating ϵ and η by an S^7 is also allowed.

Now, Let us show what will happen to $spin(1,9)$ when we transform it to $soft\ spin(1,9)$

$$\begin{aligned} soft\ spin(1,9) \quad &\sim \quad [\Gamma_i, \Gamma_j] \, \eta \\ &= \left[\begin{pmatrix} 0 & \mathbb{E}_i \\ -\mathbb{E}_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{E}_j \\ -\mathbb{E}_j & 0 \end{pmatrix} \right] \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \\ &= - \begin{pmatrix} 0 & [\mathbb{E}_i, \mathbb{E}_j] \\ [\mathbb{E}_i, \mathbb{E}_j] & 0 \end{pmatrix} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \\ &= - \begin{pmatrix} 0 & f_{ijk}^{(+)} \left(\varphi^{(2)} \right) \mathbb{E}_k \\ f_{ijk}^{(+)} \left(\varphi^{(1)} \right) \mathbb{E}_k & 0 \end{pmatrix} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}. \end{aligned} \tag{42}$$

5 Some Superspace Hints

Lastly, let us make some comments about a possible superspace. It seems that the best way to find the $d = 6, 10$ superspace for SSYM is by defining some quaternionic and octonionic Grassmann variables that decompose the corresponding spinors into an $SL(2, H)$ and an $SL(2, soft\ S^7)$ respectively

$$\{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = 0, \tag{43}$$

where $\alpha = 1, 2$ over quaternions or octonions. We know that the supersymmetry generators Q_α are derived from right multiplication

$$Q_\alpha = \left(\partial_\alpha - 1|\Gamma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} P_\mu \right) \tag{44}$$

$$Q^\alpha = \left(-\partial^\alpha + \bar{\theta}_{\dot{\beta}} 1|\tilde{\Gamma}^{\mu\dot{\beta}\alpha} P_\mu \right) \tag{45}$$

also

$$\bar{Q}^{\dot{\alpha}} = \left(\partial^{\dot{\alpha}} - 1|\tilde{\Gamma}^{\mu\dot{\alpha}\alpha}\theta_\alpha P_\mu \right) \tag{46}$$

$$\bar{Q}_{\dot{\alpha}} = (-\partial_{\dot{\alpha}} + \theta^\alpha 1|\Gamma_{\alpha\dot{\alpha}} P_\mu) \tag{47}$$

whereas the covariant derivative D_α are obtained by left action

$$D_\alpha = \left(\partial_\alpha + \Gamma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} P_\mu \right) \tag{48}$$

$$D^\alpha = \left(-\partial^\alpha - \bar{\theta}_{\dot{\beta}}\tilde{\Gamma}^{\mu\dot{\beta}\alpha} P_\mu \right) \tag{49}$$

also

$$\bar{D}^{\dot{\alpha}} = \left(\partial^{\dot{\alpha}} + \tilde{\Gamma}^{\mu\dot{\alpha}\alpha}\theta_\alpha P_\mu \right) \tag{50}$$

$$\bar{D}_{\dot{\alpha}} = (-\partial_{\dot{\alpha}} - \theta^\alpha \Gamma_{\alpha\dot{\alpha}} P_\mu) \tag{51}$$

Leading to a result acceptable but different from the standard $N = 1$, $d = 4$ superspace,

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2\left(1|\Gamma^\mu_{\alpha\dot{\alpha}}\right) P_\mu \,,$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \,,$$

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2\Gamma^\mu_{\alpha\dot{\alpha}} P_\mu \,,$$
$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \,,$$

and iff left and right action commute, we restore

$$\{Q_\alpha, \bar{D}_{\dot{\alpha}}\} = \{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = 0 \,,$$
$$\{Q_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \,.$$

On the other hand for octonions we would have the weaker conditions,

$$\begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix} \{Q_\alpha, \bar{D}_{\dot{\alpha}}\} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} = \begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix} \{D_\alpha, \bar{Q}_{\dot{\alpha}}\} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} = 0 \,,$$

$$\begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix} \{Q_\alpha, D_\beta\} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} = \begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix} \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} = 0 \,.$$

The commutation of left and right actions is not just needed for associativity but for the invariance under supersymmetry transformation

$$\delta_\xi \equiv \xi Q + \bar{\xi}\bar{Q} \tag{52}$$

12

because only the associativity ensures

$$\begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix} [\delta_\xi, D_\alpha] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix} [\delta_\xi, \bar{D}_{\dot{\alpha}}] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \quad (53)$$

since δ_ξ is left action and D_α is right action which is a very important relation in the standard $N = 1$ superspace for the invariance of the Lagrangian under supersymmetry transformation. We hope to return to this point in a future work.

13

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